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On close-to- α -concave functions.

By

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1. Introduction.

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{ z : |z| < 1 \}$ and univalent in U.

A function $f(z) \in A$ is said to be an α-concave function, if for arbitrary two points z_1 and z_2 (z_1 and $z_2 \in U$), there exists a circular arc C which connects the points $g(z_1)$ and $g(z_2)$, contained in $g(U)$, and whose central angle is not large than $\alpha\pi$, or there exists a point z for which

$$\left| \arg \left(\frac{f(z) - f(z_1)}{f(z_2) - f(z)} \right) \right| \le \frac{\pi}{2}\alpha$$

and the line segments $\overline{f(z_1)f(z)}$ and $\overline{f(z)f(z_2)}$ are contained in $f(U)$.

Definition. A function $f(z) \in A$ is said to be close-to-α-concave, if there exists an α-concave function $g(z)$ for which $f(z)$ satisfies the condition

$$\left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\pi}{2}(1-\alpha) \quad \text{in } U$$

where $0 \leq \alpha < 1$.

2. Main theorem.

Theorem. If $f(z)$ is a close-to-α-concave function, then $f(z)$ is univalent in U.

Proof. Let z_1 and z_2 are arbitrary two points in U.

Then, from the assumption, either $g(z_1)$ and $g(z_2)$ can be connected by a circular arc C whose central angle is not larger than $\alpha\pi$ and $C \subset f(U)$ or there exists a point $z \in U$ such that

$$\left| \arg \frac{f(z) - f(z_1)}{f(z_2) - f(z)} \right| < \frac{\pi}{2}\alpha ,$$

(1) The first case, $g(z_1)$ and $g(z_2)$ can be connected by circular arc C whose central angle is not larger than $\alpha\pi$, then $g(z)$ is univalent in U and so, there exists the inverse function

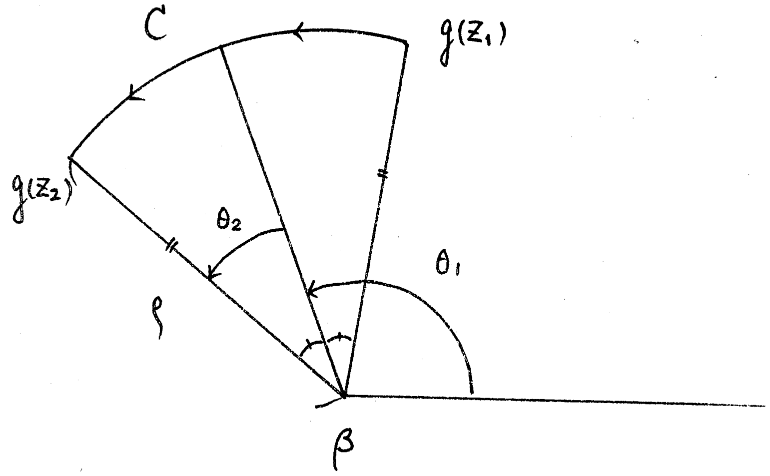
$$z = g^{-1}(\zeta).$$

Let z_1 and z_2 are arbitrary two points of U and

$$\zeta_i = g(z_i), \quad i = 1, 2.$$

Then we have

$$\begin{aligned} f(z_2) - f(z_1) &= f(g^{-1}(\zeta_2)) - f(g^{-1}(\zeta_1)) \\ &= \int_C \frac{df(g^{-1}(\zeta))}{d\zeta} d\zeta \\ &= \int_{-\theta_2}^{\theta_2} \frac{\frac{df(z)}{dz}}{\frac{d\zeta}{dz}} i\rho e^{i(\theta_1+\theta)} d\theta \end{aligned}$$



Where C is a circular arc with center β and radius ρ such that

$$\zeta = \beta + \rho e^{i(\theta_1 + \theta)},$$

$$-\frac{\pi}{2}\alpha \le -\theta_2 \le \theta \le \theta_2 \le \frac{\pi}{2}\alpha,$$

$$\zeta_1 = \beta + \rho e^{i(\theta_1 - \theta_2)},$$

and

$$\zeta_2 = \beta + \rho e^{i(\theta_1 + \theta_2)}.$$

Then we have

$$\frac{f(g^{-1}(\zeta_2)) - f(g^{-1}(\zeta_1))}{i\rho e^{i\theta_1}} = \int_{-\theta_2}^{\theta_2} \frac{f'(z)}{g'(z)} e^{i\theta} d\theta$$

Now then, we have

$$\left| \arg \frac{f'(z)}{g'(z)} e^{i\theta} \right|$$

$$\le \left| \arg \frac{f'(z)}{g'(z)} \right| + |\theta|$$

$$< \frac{\pi}{2}(1-\alpha) + \frac{\pi}{2}\alpha = \frac{\pi}{2}$$

and therefore, we have

$$f(z_1) \ne f(z_2).$$

(2) The second case, then there exists a point $z_3 \in U$ such that

$$\left| \arg \frac{g(z_3) - g(z_1)}{g(z_2) - g(z_3)} \right| \leq \frac{\pi}{2}\alpha$$

and the line segments $\overline{g(z_1)\, g(z_3)}$ and $\overline{g(z_3)\, g(z_2)}$ are contained

in $g(\mathbb{U})$ and then it follows that

$$f(z_2) - f(z_1) = (f(z_2) - f(z_3)) + (f(z_3) - f(z_1))$$

$$= \int_{l_2} \frac{d f(g^{-1}(\zeta))}{d\zeta} d\zeta + \int_{l_1} \frac{d f(g^{-1}(\zeta))}{d\zeta} d\zeta = I \text{ say},$$

where l_1 is the line segment from ζ_1 to $\zeta_3 = g(z_3)$ and l_2 is also the

line segment from $\zeta_3 = g(z_3)$ to ζ_2.

Then we have

$$I = \int_0^1 \frac{f'(z)}{g'(z)} (\zeta_3 - \zeta_1)\, dt + \int_0^1 \frac{f'(z)}{g'(z)} (\zeta_2 - \zeta_3)\, dt$$

and so, it follows that

$$\frac{f(z_2) - f(z_1)}{\zeta_3 - \zeta_1} = \int_0^1 \frac{f'(z)}{g'(z)} dt + \int_0^1 \frac{f'(z)}{g'(z)} \left(\frac{\zeta_2 - \zeta_3}{\zeta_3 - \zeta_1} \right) dt .$$

Then, from the assumption, we have

$$\left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\pi}{2}(1-\alpha),$$

$$\left| \arg \frac{f'(z)}{g'(z)} \left(\frac{\zeta_2 - \zeta_3}{\zeta_3 - \zeta_1} \right) \right|$$

$$\leq \left| \arg \frac{f'(z)}{g'(z)} \right| + \left| \arg \left(\frac{g(z_2) - g(z_3)}{g(z_3) - g(z_1)} \right) \right|$$

$$< \frac{\pi}{2}(1-\alpha) + \frac{\pi}{2}\alpha = \frac{\pi}{2}.$$

This shows that

$$\mathrm{Re}\left(\frac{f(z_2) - f(z_1)}{\zeta_3 - \zeta_1} \right) > 0$$

and so

$$f(z_2) \neq f(z_1).$$

This completes the proof.

Remark. It is trivial that if $f(z) \in A$ satisfies

$$1 + \mathrm{Re}\, \frac{z f''(z)}{f'(z)} > -\frac{\alpha}{2} \quad \text{in } U$$

where $0 \leq \alpha < 1$, then $f(z)$ is an α-concave function.